8.5 Nonhomogeneous Equations and Undetermined coefficients

1. $f(x) =$ polynomial of deg m, $y_p = A_m x^m + A_{m-1} x^{m-1} + \dots + A_1 x + A_0$

2. $f(x) = a\cos kx$ or $b\sin kx$
   or
   $a\cos kx + b\sin kx$, $y_p = A\cos kx + B\sin kx$

3. $f(x) = a e^{kx}$, $y_p = Ae^{kx}$, A, B, A_i's are undetermined coeff.

Ex 1 $y'' + 3y' + 4y = 3x + 2$

take $y_p(x) = Ax + B$

$y_p' = A$, $y_p'' = 0$

$\Rightarrow$ $0 + 3A + 4(Ax + B) = 3x + 2 \Rightarrow A = \frac{3}{4}$, $B = -\frac{1}{16}$

$y_p(x) = \frac{3}{4}x - \frac{1}{16}$.

Ex 2. $y'' - 4y = 2e^{3x}$

take $y_p(x) = Ae^{3x}$

$\Rightarrow$ $9Ae^{3x} - 4e^{3x}A = 2e^{3x}$

$\Rightarrow$ $5A = 2 \Rightarrow A = 2/5 \Rightarrow y_p = \frac{2}{5}e^{3x}$

Ex 3. $3y'' + y' - 2y = 2\cos x$

$$y_p = A\cos x + B\sin x$$

$$\Rightarrow 3(-A\cos x - B\sin x) + (-A\sin x + B\cos x) - 2(A\cos x + B\sin x) = 2\cos x$$

$$\Rightarrow \begin{array}{l} -5A + B = 2 \\ -A - 5B = 0 \end{array} \Rightarrow A = \frac{-5}{13}, \quad B = \frac{1}{13}.$$

$$y_p = \frac{-5}{13}\cos x + \frac{1}{13}\sin x$$

<u>Ex 4</u> $\quad y'' - 4y = 2e^{2x}$

take $y_p = Ae^{2x}$

$$\Rightarrow 4Ae^{2x} - 4Ae^{2x} = 0 \neq 2e^{2x}$$

take $y_p(x) = Axe^{2x}$

$$y_p' = Ae^{2x} + 2Axe^{2x}, \quad y'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}$$

$$\Rightarrow 4Axe^{2x} + 4Ae^{2x} - 4Axe^{2x} = 2e^{2x}$$

$$\Rightarrow 4A = 2, \quad A = \frac{1}{2}$$

$$y_p(x) = \frac{1}{2}xe^{2x}$$

<u>Rule 1</u> : To find particular ~~f~~ soln. of $Ly = f(x)$ with constt coeff and $f(x)$ is l. c. of finite products of fns of type 1, 2 or 3.

Suppose no term in $f(x)$ or any of its derivatives satisfy $Ly = 0$. Then take trial fn for y_p a l. c. of all l. i terms in f and their derivatives.

Ex 5 Find a particular soln of

$$y'' + 4y = 3x^3$$

Ch. eqn $(r^2 + 4) = 0 \Rightarrow r = \pm 2i$

$$y_c = C_1 \cos 2x + C_2 \sin 2x$$

By rule 1 $\quad y_p = Ax^3 + Bx^2 + Cx + D$

$$y_p' = 3Ax^2 + 2Bx + C, \quad y_p'' = 6Ax + 2B$$

$$\Rightarrow 6Ax + 2B + 4(Ax^3 + Bx^2 + Cx + D) = 3x^3$$

$$\Rightarrow 4A = 3, \quad 4B = 0, \quad 6A + 4C = 0, \quad 2B + 4D = 0$$

$$\Rightarrow A = \frac{3}{4}, \quad B = 0, \quad C = -\frac{9}{8}, \quad D = 0$$

$$\therefore y_p = \frac{3}{4}x^3 - \frac{9}{8}x$$

Ex 6 Solve I.V.P

$$y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x, \quad y(0) = 1, \; y'(0) = 2$$

Ch. eqn. $\quad r^2 - 3r + 2 = 0 \Rightarrow (r-1)(r-2) = 0$

$$\Rightarrow r = 1, 2.$$

$$y_c = C_1 e^{x} + C_2 e^{2x}$$

$$\therefore y_p = Ae^{-x} + B\cos 3x + C\sin 3x$$

$$y_p' = -Ae^{-x} - 3B\sin 3x + 3C\cos 3x$$

$$y_p'' = Ae^{-x} - 9B\cos 3x - 9C\sin 3x$$

$\Rightarrow A = \frac{1}{2},\ B = \frac{7}{13},\ C = \frac{9}{13}$

$\therefore$ general soln

$$y(x) = y_c + y_p$$

$$= c_1 e^{x} + c_2 e^{2x} + \frac{1}{2} e^{-x} + \frac{7}{13} \cos 3x + \frac{9}{13} \sin 3x .$$

$$y'(x) = c_1 e^{x} + 2c_2 e^{2x} - \frac{1}{2} e^{-x} - \frac{7\times 3}{13} \sin 3x + \frac{27}{13} \cos 3x$$

$$y(0) = c_1 + c_2 + \frac{1}{2} + \frac{7}{13} = 0$$

$$y'(0) = c_1 + 2c_2 - \frac{1}{2} + \frac{27}{13} = 2$$

$$\Rightarrow c_1 = -\frac{1}{2},\ c_2 = \frac{6}{13}$$

$$\therefore y(x) = -\frac{1}{2} e^{x} + \frac{6}{13} e^{2x} + \frac{1}{2} e^{-x} + \frac{7}{13} \cos 3x + \frac{9}{13} \sin 3x .$$

<u>Ex 7.</u> Find general form of particular soln of

$$y^{(3)} + 9y' = x \sin x + x^2 e^{2x}$$

$$(r^3 + 9r) = 0 \Rightarrow r(r^2 + 9) \Rightarrow r(r \pm 3i) = 0$$

$$\Rightarrow r = 0,\ r = 3i,\ r = -3i$$

$$\therefore y_c = c_1 + c_2 \cos 3x + c_3 \sin 3x$$

$x \sin x,\ \sin x,\ x \cos x,\ \cos x,\ x^2 e^{2x},\ x e^{2x},\ e^{2x}$

$$y_p(x) = A \cos x + B \sin x + Cx \cos x + Dx \sin x + E e^{2x} + F x e^{2x} + G x^2 e^{2x} .$$

The case of Duplication ie when terms of f or derivatives already occur in y_c.

Rule 2 For $f(x) = P_m(x) e^{rx} \cos kx$ or $P_m(x) e^{rx} \sin kx$ take the trial soln

$$y_p(x) = x^s \left[(A_0 + A_1 x + \cdots + A_m x^m) e^{rx} \cos kx + (B_0 + B_1 x + \cdots + B_m x^m) e^{rx} \sin kx \right]$$

where s is the smallest non-ve integer ~~n~~ term in y_p duplicates a term in y_c.

Ex8 Find y_p of $y^{(3)} + y'' = 3e^x + 4x^2$

$r^3 + r^2 = 0 \Rightarrow r^2(r+1) = 0 \quad r = 0, 0, -1$

$y_c(x) = C_1 + C_2 x + C_3 e^{-x}$

$$y_p = Ae^x + (B + Cx + Dx^2) \cdot x^2$$

$$= Ae^x + Bx^2 + Cx^3 + Dx^4$$

$$y_p' = Ae^x + 2Bx + 3Cx^2 + 4Dx^3$$

$$y_p'' = Ae^x + 2B + 6Cx + 12Dx^2$$

$$y_p^{(3)} = Ae^x + 6C + 24Dx$$

$\Rightarrow Ae^x + 6C + 24Dx + Ae^x + 2B + 6Cx + 12Dx^2 = 3e^x + 4x^2$

$\Rightarrow A = 3/2, \quad B = 4, \quad C = -\frac{4}{3}, \quad D = \frac{1}{3}$

$$y_p = \frac{3}{2}e^x + 4x^2 - \frac{4}{3}x^3 + \frac{1}{3}x^4$$

Ex 9 $$y'' + 6y' + 13y = e^{-3x}\cos 2x$$

$$r^2 + 6r + 13 = 0 \Rightarrow r = -3 \pm 2i$$

$$y_c = e^{-3x}(C_1\cos 2x + C_2\sin 2x)$$

$$y_p = e^{-3x}(A\cos 2x + B\sin 2x) \times x \qquad 1)$$

Ex 10. $$(D-2)^3(D^2+9)y = x^2e^{2x} + x\sin 3x$$

$$r = 2, 2, 2 \pm 3i$$

$$y_c = (C_1 + C_2x + C_3x^2)e^{2x} + C_4\cos 3x + C_5\sin 3x$$

$$y_p = \left[Ax^2e^{2x} + Bxe^{2x} + Ce^{2x}\right] \times x^3 + [Dx\sin 3x + Ex\cos 3x + F\sin 3x + G\cos 3x] \times x$$

$$\Rightarrow y_p = Ax^5e^{2x} + Bx^4e^{2x} + Cx^3e^{2x} + Dx^2\sin 3x + Ex^2\cos 3x + Fx\sin 3x + Gx\cos 3x.$$

Method of Variation of parameters.

$$Ly = y'' + p(x)y' + Q(x)y = f(x)$$

$$y_c = C_1y_1(x) + C_2y_2(x)$$

take $$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \quad \text{—— } (*)$$

$$y_p' = (u_1y_1' + u_2y_2') + (u_1'y_1 + u_2'y_2)$$

where u_1, u_2 are fns to be determined s.t.

$$L y_p = f(x) \quad \text{— (1)}$$

& $$u_1' y_1 + u_2' y_2 = 0 \quad \text{— (2)}$$ (so that there is no u_1'', u_2'')

$$\therefore \; y_p' = u_1 y_1' + u_2 y_2' \quad \text{— (3)}$$

$$y_p'' = (u_1 y_1'' + u_2 y_2'') + (u_1' y_1' + u_2' y_2')$$

Since y_1, y_2 are solns of homo eqn

$$y'' + P(x) y' + Q(x) y = 0$$

$$\Rightarrow \; y_i'' = -P y_i' - Q y_i \quad (i = 1, 2.$$

$$\therefore \; y_p'' = u_1' y_1' + u_2' y_2' - P(u_1 y_1' + u_2 y_2') - Q(u_1 y_1 + u_2 y_2)$$

$$= u_1' y_1' + u_2' y_2' - P y_p' - Q y_p \quad ((3) \; \& \; (2))$$

$$\therefore \; L y_p = u_1' y_1' + u_2' y_2' = f(x) \quad (\text{by } (1)$$

$\therefore$ u_1, u_2 are to be determined s.t

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = f(x)$$

$$u_1' = \frac{-f(x)\, y_2}{y_1 y_2' - y_1' y_2} = -\frac{y_2 f(x)}{W}, \quad u_2' = \frac{y_1 f(x)}{W}$$

$$y_p(x) = y_1 u_1 + y_2 u_2 = -y_1 \int \frac{y_2(x) f(x)\, dx}{W} + y_2(x) \int \frac{y_1(x) f(x)\, dx}{W}$$

Ex 11 $y'' + y = \tan x$, $r^2 + 1 = 0$ $r = \pm i$

$$y_c(x) = \underbrace{C_1 \cos x}_{y_1} + \underbrace{C_2 \sin x}_{y_2}$$

$$y_p = u_1 y_1 + u_2 y_2 = u_1 \cos x + u_2 \sin x$$

$$y_p' = -u_1 \sin x + u_2 \cos x + u_1' \cos x + u_2' \sin x$$

Let $u_1' \cos x + u_2' \sin x = 0$ — (1)

$$y_p'' = -u_1' \sin x + u_2' \cos x - u_1 \cos x - u_2 \sin x$$

$$y_p'' + y = -u_1' \sin x + u_2' \cos x = \tan x \quad — (2)$$

$$u_1' \cos x + u_2' \sin x = 0$$
$$-u_1' \sin x + u_2' \cos x = \tan x$$

$$u_1' = -\sin x \tan x = -\frac{\sin^2 x}{\cos x} = -\sec x + \cos x$$
$$u_2' = \tan x \cos x = \sin x$$

$\therefore$ $u_1 = \sin x - \ln|\sec x + \tan x|$

$$u_2 = -\cos x$$

$$y_p = (\sin x - \ln|\sec x + \tan x|) \cos x + (-\cos x) \cdot \sin x$$
$$= -\cos x \ln|\sec x + \tan x|$$

$\therefore$ $y(x) = y_c + y_p$.